1. **Divided Stock Pen Problem:** Suppose you are building a rectangular stock pen using 600 feet of fencing. You will use part of this fencing to build a fence across the middle of the rectangle. Find the length and width of the rectangle that give the maximum total area. Justify your answer by plotting the equation for area.

   \[
   \text{Perimeter: } 600 = 2x + 3y \\
   \Rightarrow y = \frac{200 - \frac{2}{3}x}{2} \\
   \Rightarrow y = \frac{200 - \frac{2}{3}(150)}{2} \\
   \Rightarrow y = 100 \\
   \Rightarrow A = 200x - \frac{2}{3}x^2 \\
   \Rightarrow A'(x) = 200 - \frac{4}{3}x \\
   \Rightarrow A'(150) = 0 \\
   \Rightarrow A = 15000
   \]

   The length of 150 ft and width of 100 ft have a max area of 15,000 ft².

2. **Two-Field Problem:** Ella Mentary has 600 feet of fencing to enclose two fields. One field will be a rectangle twice as long as it is wide, and the other will be a square. The square field must contain at least 100 ft². The rectangular one must contain at least 800 ft².
   a. If \( x \) is the width of the rectangular field, what is the domain of \( x \)?

   \[
   \text{Area Rectangle: } A = 100x + 200 \\
   \text{Area Square: } A = 2x^2 \\
   \text{Perimeter: } 600 = 4x + 2y \\
   \Rightarrow 150 \leq x \leq 93.333...
   \]

   b. Plot the graph of the total area contained in the two fields as a function of \( x \).

   \[
   A = 4.25x^2 - 450x + 22500
   \]

   c. What is the greatest area that can be contained in the two fields? Justify your answer.

   According to the graph of \( A \), the maximum area occurs at endpoint \( x = 93.333... \), which cannot be found by calculating when \( A' = 0 \).

   \[
   A(93.333...) = 4.25(93.333...)^2 - 450(93.333...) + 22500 = 17522.222...
   \]

   The maximum area is 17,522.222 ft².

3. **Two-Corral Problem:** You work on Bill Spender's Ranch. Bill tells you to build a circular fence around the lake and to use the remainder of your 1000 yards of fencing to build a square corral. To keep the fence out of the water, the diameter of the circular enclosure must be at least 50 yards.
   a. If you must use all 1000 yards of fencing, how can you build the fences to enclose the minimum total area? Justify your answer by analyzing the signs of the first derivative.

   \[
   \text{Perimeter: } P = 2\pi r + 4s \\
   1000 = 2\pi r + 4s \\
   S = 250 - \frac{1}{2} \pi r \\
   \text{Area: } A = \pi r^2 + s^2 \\
   = \pi r^2 + (250-\frac{1}{2} \pi r)^2 \\
   \text{Domain of } r: 2 \leq 50 \Rightarrow r \geq 25 \\
   \text{All fencing used.} \\
   \text{Area: } A = \pi r^2 + s^2 \\
   A'(r) = 2\pi r - 2(250-\frac{1}{2} \pi r) - \frac{1}{2} \pi \\
   0 = 2\pi r - 2(250-\frac{1}{2} \pi r) \\
   2\pi r = 2(250-\frac{1}{2} \pi r) \\
   r = 70.012...
   \]

   \[
   \text{Endpoint Check: } \\
   A(70) = 35006.197...
   \]

   \[
   \text{Sign Check: } \\
   A'(70) = -0.159... \\
   A'(71) = 11.098... \\
   \Rightarrow -0+.. = \text{local min.}
   \]
The minimum total area is 35006.197... yards², which occurs at local minimum
r = 70.012... yards because the first derivative changes from - to +. This
local minimum is also the global minimum because the area is lower
than the area at the endpoints.

b. What would you tell Bill if he asked you to build the fences to enclose the maximum total area?

The maximum area occurs at the largest possible circle. Bill should use
all 1000 yards for the circular fence and not to build a corral.

4. **Open Box I:** A rectangular box with a square base and no top is to be constructed using a total
of 120 cm² of cardboard.
a. Find the dimensions of the box of maximum volume. Justify your answer by plotting the
equation for volume.

\[
\begin{align*}
\text{Surface Area} & \quad A = 4xy + x^2 \\
120 & = 4xy + x^2 \\
y & = \frac{120 - x^2}{4x} \\
\text{Volume} & \quad V = LWH \\
& = x^2 \left(\frac{120 - x^2}{4x}\right) \\
& = 30x - \frac{1}{4}x^3 \\
\text{Domain of Volume} & \quad 0 = \frac{3}{4}(120 - x^2) \\
& = \frac{x}{4}(120 - x^2) \\
& = 0 \\
& = 120 - x^2 = 0 \\
x & = \pm \sqrt{120} \Rightarrow \text{For positive values } 0 \leq x \leq \sqrt{120}
\end{align*}
\]

\[
\begin{align*}
V' & = 0 \\
V' & = 30 - \frac{3}{4}x^2 \\
0 & = 30 - \frac{3}{4}x^2 \\
\frac{3}{4}x^2 & = 30 \\
x^2 & = \frac{40}{3} \\
x & = \pm \sqrt{\frac{40}{3}} \\
x & = \frac{2}{3}\sqrt{30} \\
it & = 6.324... \\
\text{Max Vol.} & = 126.491... \text{ cm}^3
\end{align*}
\]

\[
\begin{align*}
\text{Base length or width:} & \quad 2\sqrt{10} \text{ or } 6.324... \text{ cm} \\
\text{Height:} & \quad 3.162... \text{ cm} \\
\text{Max Vol.:} & \quad 126.491... \text{ cm}^3
\end{align*}
\]

5. **Open Box III:** You are building a glass fish tank that will hold 72 ft³ of water. You want its base and sides to be
rectangular and the top, of course, to be open. You want to construct the tank so that its width is 5 feet but the
length and depth are variable. Building materials for the tank cost $10 per square foot for the base and $5 per
square foot for the sides. What is the cost of the least expensive tank? Justify your answer by analyzing the signs of
the first derivative.

\[
\begin{align*}
\text{Volume} & \quad V = LWH \\
72 & = 5x \cdot y \\
5x & = \frac{72}{y} \\
5x & = \frac{72}{y} \\
5x & = \frac{360}{x} \\
\text{Surface Area and Price} \\
\text{Base:} & \quad 50x \\
& = 50 \times 5 \\
& = 250 \\
\text{Side:} & \quad 5 \cdot y \cdot 5 \\
& = 5 \cdot \frac{72}{5} \cdot 5 \\
& = 360 \times 5 \\
\text{Price:} & \quad x \cdot y \cdot 5 \\
& = x \cdot \frac{72}{5} \cdot 5 \\
& = 72
\end{align*}
\]

\[
\begin{align*}
\text{Total Cost} & \quad C = 50x + 2\left(\frac{360}{x}\right) + 2(72) \\
& = 50x + \frac{720}{x} + 144 \\
\Rightarrow C & = (3.794...) = 523.47... \\
\text{There is a local minimum at} \\
x & = 3.794... \text{ because } C' \text{ changes from } - \text{ to } + \text{ there.} \\
The minimum cost is $523.47.
\end{align*}
\]

\[
\begin{align*}
C' & = 0 \\
C' & = 50 - 720x^{-2} \\
0 & = 50 - 720x^{-2} \\
720x^{-2} & = 50 \\
x & = \pm 3.794... \\
x & = 3.794... \\
\text{C' Sign Check} \\
C'(3.7) & = -2.593... \\
C'(3.8) & = 0.138... \\
\Rightarrow -0 + \text{ (local min)}
\end{align*}
\]
6. **Shortest-Distance Problem**: What point on the graph of \( y = e^x \) is closest to the origin? Justify your answer by plotting the equation for distance.

\[
D = \sqrt{x^2 + y^2} = \sqrt{x^2 + e^{2x}}
\]

\[
D^2 = x^2 + e^{2x}
\]

\[
D^2 = \frac{1}{2} \left( x^2 + e^{2x} \right) - \frac{1}{2} \left( 2x + 2e^{2x} \right)
\]

\[
0 = \frac{1}{2} \left( x^2 + e^{2x} \right) - \frac{1}{2} \left( 2x + 2e^{2x} \right)
\]

Cannot be solved; use calculator.

\[ x = -0.4263... \]

Closest point to the origin is \((x, y) = (-0.4263..., 0.6529...)\)

7. **Ladder Problem**: A ladder is to reach over a fence 8 feet high to a wall that is 1 foot behind the fence. What is the length of the shortest ladder that you can use? Justify your answer by plotting the equation for the length of the ladder.

\[
\frac{y}{x} = \frac{8}{x-1}
\]

\[
y = \frac{8x}{x-1}
\]

Length of ladder \( L \):

\[
L = x + \sqrt{x^2 + \left( \frac{8x}{x-1} \right)^2}
\]

or

\[
L = \sqrt{x^2 + \left( \frac{8x}{x-1} \right)^2}
\]

\[ L(5) = 11.180... \text{ ft} \]

The shortest ladder you can use is 11.180... ft.

8. **Tin Can Problem**: A popular size of tin can with “normal” proportions has diameter 7.3 cm and height of 10.6 cm.

a. What is its volume?

\[
V = \pi r^2 h = \pi \left( \frac{7.3}{2} \right)^2 \times 10.6 = 141.218... \pi = 443.651... \text{ cm}^3
\]

b. The volume is to be kept the same, but the proportions are to be changed. Write an equation expressing total surface of the can (lateral surface plus two ends) as a function of radius and height. Transform the equation so that the volume is in terms of radius alone.

\[
V = 2\pi r^2 + 2\pi rh
\]

\[
A = 2\pi r^2 + 2\pi r \left( \frac{141.218...}{r} \right)
\]

\[
A = 2\pi \left( r^2 + 141.218... r^{-1} \right)
\]

c. Find the radius and height of the can that minimize its surface area. Justify your answers by analyzing the signs of the first derivative.

\[
A' = 2\pi \left( 2r - 141.218... r^{-2} \right)
\]

\[
= 2\pi r^2 - 2(2r^3 - 141.218...) = 0
\]

\[
r = \sqrt[3]{70.609...} = 4.133...
\]

\[
h = \frac{141.218...}{r^2} = 4.133...
\]

\[
h = 8.266...
\]

The can will need a radius of 4.133... cm and height of 8.266... cm to minimize the surface area.
d. Does the normal can use close to the minimum amount of metal? What percentage of the metal in the normal can could you save by using cans with the minimum dimensions?

\[
\text{Normal Can: } A = 2\pi \left( \frac{3}{2} \right)^2 + 2\pi \left( \frac{3}{2} \right) (10.6)
\]
\[
= 326.8041... \text{ cm}^2
\]

\[
\text{Minimal Can: } A = 2\pi (4.133...)^2 + 2\pi (4.133...) (9.260...)
\]
\[
= 322.014... \text{ cm}^2
\]

\[
\text{The minimal can saves about 1.465% of metal used in a normal can.}
\]

The minimal can saves about 1.465% of metal used in a normal can.

\[
\text{Savings: (0.06) (20,000,000) (0.01465...) (365) = 564,419... million}
\]

\#17.9. *Cup Problem*: You have been hired by the Yankee Cup Company. They currently make a cylindrical paper cup of diameter 5 cm and height 7 cm. Your job is to find ways to save paper by making cups that hold the same amount of liquid but have different proportions. Find the dimensions of a same-volume cylindrical cup that uses a minimum amount of paper. Justify your answer by plotting the equation for the surface area.

Graph of \( A = 97.5\pi r^2 + \pi r^2 \)

\[
\text{Volume: } V = \pi r^2 h
\]
\[
= \pi (7.5)^2 (7)
\]
\[
= 137.75\pi
\]

\[
\text{Surface Area: } A = 2\pi rh + \pi r^2
\]
\[
= 2\pi \left( \frac{43.75}{r} \right) + \pi r^2
\]
\[
= 87.5\pi r^{-1} + \pi r^2
\]

\[
A^1 = 0
\]
\[
A^1 = -87.5\pi r^{-2} + 2\pi r
\]
\[
0 = \pi r^{-2} (-87.5 + 2r^3)
\]
\[
-87.5 + 2r^3 = 0
\]
\[
2r^3 = 87.5
\]
\[
\frac{2r^3}{2} = \frac{87.5}{2}
\]
\[
r = 3.523...
\]

Cups with radius of 3.523... cm and height of 3.523... cm have a minimum surface area of 137.444... cm².