8.2: Critical Points & Points of Inflection

**Objective:** From information about the first and second derivatives of a function, decide whether the y-value is a local maximum or minimum at a critical point and whether the graph has a point of inflection, then use this information to sketch the graph or find the equation of the function.

### Critical Points (First Derivative Analysis)

The critical point(s) of a function is the x-value(s) at which the first derivative is zero or undefined. The y-value of a critical point may be classified as a local (relative) minimum, local (relative) maximum, or a plateau point.

**Types of Critical Points**

![Graph of a function showing critical points](image)

Although you can classify each type of critical point by seeing the graph, you can draw a number-line to analyze the behavior around each critical point and justify your classification of each critical point.

- The critical points are \( x_1, x_2, x_3, \) and \( x_4 \) because the first derivative of the function equals zero or is undefined at those x-values.
- There is a local (relative) maximum at \( x_1 \) and \( x_3 \) because the first derivative changes signs from positive to negative.
- There is a local (relative) minimum at \( x_2 \) because the first derivative changes signs from negative to positive.
- The critical point, \( x_4 \), is not a maximum or minimum because the first derivative’s signs remains negative (unchanged).

**Notes:**

* Critical points are graphical features of an original function. Analyzing the derivative of the function helps you identify the critical points.
* If there were no endpoints and \( f'(x) \) continued going down, then there would be no definite global (absolute) minimum.
* Global (absolute) maximum and minimums are not critical points because the first derivative is not equal to zero or undefined at those points.

\[
\begin{array}{c|c|c|c|c}
\text{x} & \text{max} & \text{min} & \text{cusp} & \text{plateau} \\
\hline
x_1 & 0 & 0 & \text{Und.} & 0 \\
\end{array}
\]
**Inflection Points (Second Derivative Analysis)**

The *inflection point(s)* of a function is the *x*-value(s) at which the second derivative is zero or undefined and the function is changing concavity. You can tell that the function changes concavity if the second derivative changes signs.

- The inflection points are $x_3$ and $x_4$ because the second derivative of the function equals zero or is undefined at those $x$-values, and the sign of the second derivative changes signs:
  - At $x_5$ the second derivative changes signs from negative to positive, which means the function changes concavity from concave down to concave up.
  - At $x_4$ the second derivative changes signs from positive to negative, which means the function changes concavity from concave up to concave down.
- Although the second derivative is undefined at $x_3$, it is not an inflection point because the second derivative does not change signs, it remains concave up.

**Other Notes on Concavity:**

- **Und.**
- **p.i.**
- **no p.i.**
- **p.i.**

Sketch of Second Derivative:

- $y = f(x)$
- $f''(x)$
- $x_1$, $x_5$, $x_2$, $x_3$, $x_4$
**Example 1: Critical & Inflection Points Given a Graph of $f$**

For the graphed function, sketch a number-line graph for $f'$ and a number-line graph for $f''$ that shows the sign of each derivative in a neighborhood of the critical point at $x = 2$. On the number-line graphs, indicate whether $f(2)$ is a local maximum or a local minimum and whether the graph has a point of inflection at $x = 2$.

**Example 2: Critical Points & Points of Inflection Given a Number-Line of Derivatives**

The figure shows number-line graphs for the first and second derivatives of a continuous function $f$. Use this information to sketch the graph of $f$ if $f(4) = 0$. Describe the behavior of the function at the critical points.
Example 3: Critical & Inflection Points Given a Graph of $f'$

The figure shows the graph of the derivative of a continuous, piecewise function $f$ defined on the closed interval $x \in [0, 8]$. Sketch, the graph of $f$, given the initial condition that $f(1) = 5$. Put a dot at the approximate location of each critical point and each point of inflection.

Example 4: Critical & Inflection Points Given an Equation for $f$

The figure shows the graph of $f(x) = x^{4/3} + 4x^{1/3}$.

a. Sketch number-line graphs of $f'$ and $f''$ that show features that appear clearly on the graph.
b. Find equations for $f'$ and $f''$. Show algebraically that the critical points you drew in part a are correct. Fix any errors.

c. Write $x$- and $y$-coordinates of all maxima, minima, and points of inflection.

Example 5: Critical & Inflection Points Given an Equation for $f$

Let $f(x) = -x^3 + 4x^2 + 5x + 20$, with domain $x \in [-2.5, 5]$.

a. Plot the graph. Estimate the $x$- and $y$-coordinates of all local maxima or minima and of all points of inflection. State the global maximum and minimum.
b. Write equations for $f'(x)$ and $f''(x)$. Use them to find, either numerically or algebraically, the precise values of the $x$-coordinates in part a.

c. Show that the second derivative is negative at the local maximum point and positive at the local minimum point. Explain the graphical meaning of these facts.

d. Explain why there are no other critical points or points of inflection.
DEFINITIONS: Critical Points and Related Features

- A critical point on a graph occurs at \( x = c \) if and only if \( f(c) \) is defined and \( f'(c) \) either is zero or is undefined.

- \( f(c) \) is a local maximum (or relative maximum) of \( f(x) \) if and only if \( f(c) \geq f(x) \) for all \( x \) in a neighborhood of \( c \) (that is, in an open interval containing \( c \)).

- \( f(c) \) is a local minimum (or relative minimum) of \( f(x) \) if and only if \( f(c) \leq f(x) \) for all \( x \) in a neighborhood of \( c \).

- \( f(c) \) is the global maximum (or absolute maximum) of \( f(x) \) if and only if \( f(c) \geq f(x) \) for all \( x \) in the domain of \( f \).

- \( f(c) \) is the global minimum (or absolute minimum) of \( f(x) \) if and only if \( f(c) \leq f(x) \) for all \( x \) in the domain of \( f \).

- The graph of \( f \) is concave up at \( x = c \) if and only if for all \( x \) in a neighborhood of \( c \), the graph lies above the tangent line at the point \((c,f(c))\). The graph of \( f \) is concave down at \( x = c \) if and only if for all \( x \) in a neighborhood of \( c \), the graph lies below the tangent line at \( x = c \). The value of \( f''(c) \) is called the concavity of the graph of \( f \) at \( x = c \).

- The point \((c,f(c))\) is a point of inflection, or inflection point, if and only if \( f''(x) \) changes sign at \( x = c \). (Old spelling: inflexion, meaning "not bent.")

- The point \((c,f(c))\) is a cusp if and only if \( f' \) is discontinuous at \( x = c \). If the concavity changes at \( x = c \), but the secant lines on both sides of \( c \) do not approach a common tangent line, the term corner is sometimes used instead of cusp.

- The point \((c,f(c))\) is a plateau point if and only if \( f'(c) = 0 \), but \( f''(x) \) does not change sign at \( x = c \).

PROPERTIES: Maximum, Minimum, and Point of Inflection

- If \( f'(x) \) goes from positive to negative at \( x = c \) and \( f \) is continuous at \( x = c \), then \( f(c) \) is a local maximum.

- If \( f'(x) \) goes from negative to positive at \( x = c \), and \( f \) is continuous at \( x = c \), then \( f(c) \) is a local minimum.

- If \( f''(c) \) is positive, then the graph of \( f \) is concave up at \( x = c \).

- If \( f''(c) \) is negative, then the graph of \( f \) is concave down at \( x = c \).

- If \( f''(x) \) changes sign at \( x = c \) and \( f \) is continuous at \( x = c \), then \((c,f(c))\) is a point of inflection (by definition).

- The second derivative test: If \( f'(c) = 0 \) and \( f''(c) \) is positive (graph concave up), then \( f(c) \) is a local minimum. If \( f'(c) = 0 \) and \( f''(c) \) is negative (graph concave down), then \( f(c) \) is a local maximum. If \( f'(c) = 0 \) and \( f''(c) = 0 \), then the second derivative test fails to distinguish among maximum, minimum, and plateau points.

- A maximum or minimum point (but not a point of inflection) can occur at an endpoint of the domain of a function.